

# PAULI-FIERZ MODEL WITH KATO-CLASS POTENTIALS AND EXPONENTIAL DECAYS

Takeru Hidaka <sup>\*</sup>and Fumio Hiroshima <sup>†</sup>

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## Abstract

Generalized Pauli-Fierz Hamiltonian with Kato-class potential  $K_{\text{PF}}$  in nonrelativistic quantum electrodynamics is defined and studied by a path measure.  $K_{\text{PF}}$  is defined as the self-adjoint generator of a strongly continuous one-parameter symmetric semigroup and it is shown that its bound states spatially exponentially decay pointwise and the ground state is unique.

## 1 Introduction

In this paper we investigate generalized Pauli-Fierz Hamiltonians with Kato-class potentials in nonrelativistic quantum electrodynamics by a path measure. It includes not only Kato-class potentials but also general cutoff functions of quantized radiation fields. Basic ingredients in this paper are path measures and functional integral representation of semigroups. It has been shown that functional integral representations are useful tools to investigate the spectrum of models in quantum field theory. See e.g., [BHLMS02, Gub06, Hir00-a, Hir07, HL08, LMS02a, Nel64, Spo98, Spo04].

The strongly continuous one-parameter semigroup  $(e^{-tH_{\text{p}}})_{t \geq 0}$  generated by the Schrödinger operator,  $H_{\text{p}} = \frac{1}{2}(p - a)^2 + V$ , on  $L^2(\mathbb{R}^d)$  with some external potential  $V$  and vector potential  $a = (a_1, \dots, a_d)$  is expressed by a path measure, which is known as Feynman-Kac-Itô formula [Sim79]:

$$(f, e^{-tH_{\text{p}}}g) = \int dx \bar{f}(x) \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds - i \int_0^t a(B_s) \circ dB_s} g(B_t) \right], \quad (1.1)$$

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<sup>\*</sup>Faculty of Mathematics, Kyushu University, Fukuoka 819-0385, Japan.

<sup>†</sup>Faculty of Mathematics, Kyushu University, Fukuoka 819-0385, Japan.

where  $\mathbb{E}^x$  denotes the expectation value with respect to the Wiener measure  $P^x$ ,  $(B_t)_{t \geq 0}$  the  $d$ -dimensional Brownian motion and  $\int_0^t a(B_s) \circ dB_s$  a Stratonovich integral.

Conversely since a Kato-class potential  $V$  satisfies that

$$\sup_x \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} \right] < \infty, \quad t \geq 0, \quad (1.2)$$

the family of mappings  $S_t$  defined by

$$S_t g(x) = \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds - i \int_0^t a(B_s) \circ dB_s} g(B_t) \right], \quad t \geq 0, \quad (1.3)$$

turns to be the strongly continuous one-parameter symmetric semigroup for a Kato-class potential  $V$ . The Schrödinger operator with a Kato-class potential  $V$  is then defined as the self-adjoint generator of  $(S_t)_{t \geq 0}$ . See e.g., [BHL00, Sim82, HIL09]. Three-dimensional Kato class includes a singular external potential such as  $V(x) = -|x|^{-a}$ ,  $0 \leq a < 2$ .

We extend this to the Pauli-Fierz Hamiltonian. The Pauli-Fierz Hamiltonian  $H_{\text{PF}}$  is a self-adjoint operator defined on the tensor product of Hilbert spaces:

$$\mathcal{H} = L^2(\mathbb{R}^d) \otimes L^2(Q), \quad (1.4)$$

where  $L^2(Q)$  is an  $L^2$ -space over a probability space  $(Q, \mathcal{B}, \mu)$  with a Gaussian measure  $\mu$ , and it describes the Schrödinger representation of the standard Boson Fock space. The Pauli-Fierz Hamiltonian  $H_{\text{PF}}$  is given by

$$H_{\text{PF}} = \frac{1}{2}(p \otimes 1 + \sqrt{\alpha} \mathcal{A})^2 + V \otimes 1 + 1 \otimes H_{\text{f}}(m), \quad (1.5)$$

where  $\alpha \geq 0$  is a coupling constant,  $H_{\text{f}}(m)$  the free field Hamiltonian with a field mass  $m \geq 0$  and  $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_d)$  a quantized radiation field with a cutoff function. See Section 2 for the detail of notations. Under some conditions on cutoff functions and  $V$  it is proven that (1.5) is self-adjoint and  $e^{-tH_{\text{PF}}}$  is then defined by the spectral resolution. In [Hir97],  $(F, e^{-tH_{\text{PF}}} G)$  is also presented by a path measure:

$$(F, e^{-tH_{\text{PF}}} G) = \int dx (F(x), (T_t G)(x))_{L^2(Q)}, \quad (1.6)$$

where  $T_t$  is of the form

$$T_t f(x) = \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} J_0^* e^{i\sqrt{\alpha} \mathcal{A}_E(K_t)} J_t G(B_t) \right] \in L^2(Q) \quad (1.7)$$

for each  $x \in \mathbb{R}^d$ . Compare with (1.3) and see (2.47) for the detail.

Our construction of generalized Pauli-Fierz Hamiltonians is closed to the procedure to define the Schrödinger operator with Kato-class potentials. We believe however that it is worthwhile extending it to the Pauli-Fierz Hamiltonian from mathematical point of view. It will be shown that the family of operators  $T_t : \mathcal{H} \rightarrow \mathcal{H}$ ,  $t \geq 0$ , can be also defined for Kato class potentials  $V$  and general cutoff functions in  $\mathcal{A}$ , and the generalized Pauli-Fierz Hamiltonian  $K_{\text{PF}}$  is defined as the self-adjoint generator of  $(T_t)_{t \geq 0}$ . Of course under some conditions  $K_{\text{PF}}$  coincides with  $H_{\text{PF}}$ , but  $K_{\text{PF}}$  permits to include more singular  $V$ 's and general cutoff functions in  $\mathcal{A}$ .

Cutoff functions of  $\mathcal{A}_\mu(x)$ ,  $\mu = 1, 2, 3$ , of the standard Pauli-Fierz Hamiltonian in three-dimension are of the form

$$e^{-ikx} e_\mu(k, j) \hat{\varphi}(k) / \sqrt{|k|} \quad (1.8)$$

with some function  $\hat{\varphi}$  and polarization vectors  $e(k, j) = (e_1(k, j), e_2(k, j), e_3(k, j))$ ,  $j = 1, 2$ . In [GHPS09] the so called Nelson model on a pseudo Riemannian manifold is studied by a path measure. Generalized Pauli-Fierz Hamiltonians include a mathematical analogue of the Nelson model on a pseudo Riemannian manifold, which is unitarily transformed to the Pauli-Fierz Hamiltonian with a variable mass. Cutoff function of the Pauli-Fierz Hamiltonian with a variable mass  $v$  is (1.8) with  $e^{ikx}$  and  $e_\mu(k, j) \hat{\varphi}(k)$  replaced by  $\Psi(k, x)$  and  $\hat{\phi}_\mu^j(k)$ , respectively:

$$\overline{\Psi(k, x)} \hat{\phi}_\mu^j(k) / \sqrt{|k|}. \quad (1.9)$$

Here  $\hat{\phi}_\mu^j(k)$  is some function and  $\Psi(k, x)$ ,  $k \neq 0$ , is the unique solution of the Lippman-Schwinger equation [Ike60]:

$$\Psi(k, x) = e^{+ikx} - \frac{1}{4\pi} \int \frac{e^{i|k||x-y|} v(y)}{|x-y|} \Psi(k, y) dy. \quad (1.10)$$

The main results of the present paper are as follows:

- (1) we define the generalized Pauli-Fierz Hamiltonian  $K_{\text{PF}}$  with Kato class potentials and generalized cutoff functions, i.e., we prove that  $(T_t)_{t \geq 0}$  is a strongly continuous one-parameter symmetric semigroup;
- (2)  $K_{\text{PF}}$  is an extension of  $H_{\text{PF}}$ ;
- (3) bound states of  $K_{\text{PF}}$  spatially exponentially decay *pointwise* and the ground is unique if it exists.

We explain an outline of (1)-(3) above.

First we define the strongly continuous one-parameter symmetric semigroup  $(T_t)_{t \geq 0}$  with Kato-class potentials and general cutoff functions by functional integral representations. Then  $K_{\text{PF}}$  is defined by  $T_t = e^{-tK_{\text{PF}}}$  for  $t \geq 0$ . We introduce two assumptions, Assumptions 2.1 and 2.12, on cutoff functions of  $\mathcal{A}$ . The former is stronger than the later. One advantage to define the generalized Pauli-Fierz Hamiltonian by a path measure is that we need only a weak condition on cutoff functions (Assumption 2.12) and external potentials. Then for arbitrary  $\alpha \in \mathbb{R}$ , Kato-class potential  $V$  and cutoff function  $\hat{\rho}_\mu^j(x, k)$  satisfying  $\hat{\rho}_\mu^j(x, k) \in C_b^1(\mathbb{R}_x^d, L^2(\mathbb{R}_k^d))$ , we can define  $K_{\text{PF}}$  as a self-adjoint operator.

Secondly we can show that

$$\frac{1}{2}(p \otimes 1 + \sqrt{\alpha}\mathcal{A})^2 \dot{+} V_+ \otimes 1 \dot{-} V_- \otimes 1 + 1 \otimes H_f(m) \quad (1.11)$$

is well defined for  $V_\pm$  such that  $0 \leq V_+ \in L_{\text{loc}}^1(\mathbb{R}^d)$  and  $0 \leq V_-$  is relatively form bounded with respect to  $p^2/2$  with a relative bound strictly smaller than one. It is shown that  $K_{\text{PF}} = (1.11)$  under Assumption 2.1 on cutoff functions.

Finally it is shown that bound states of  $K_{\text{PF}}$  spatially exponentially decays *pointwise*. To show the spatial exponential decay of bound states is very important to study the properties of spectrum of Pauli-Fierz type models. In [BFS99, GLL01, Gri01] the spatial exponential decay of bound states is shown but our method is completely different from them. Since  $\varphi_b(x) = e^{tE} e^{-tK_{\text{PF}}} \varphi_b$  for  $\varphi_b$  such that  $K_{\text{PF}} \varphi_b = E \varphi_b$ , exponential decay of  $\varphi_b(x)$  is proven by means of showing  $\sup_x \|\varphi_b(x)\|_{L^2(Q)} < \infty$  and estimating  $e^{tE} \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} \right]$ . We conclude that

$$\|\varphi_b(x)\|_{L^2(Q)} \leq D e^{-C|x|^\beta} \quad (1.12)$$

almost everywhere  $x \in \mathbb{R}^d$ , and constants  $D$  and  $C$  are independent of the field mass  $m$ . Here the exponent  $\beta$ ,  $\beta \geq 1$ , is determined by the behavior of external potential  $V$ . When  $\liminf_{|x| \rightarrow \infty} V(x) < E$ , we can take  $\beta = 1$ , and when  $V(x) = |x|^{2n}$ ,  $\beta = n + 1$  is obtained. See Theorem ?? for the detail. Furthermore from a standard argument [Hir00-a] it follows that the transformed operator  $e^{i(\pi/2)N} T_t e^{-i(\pi/2)N}$  is a positivity improving semigroup, where  $N$  denotes the number operator in  $L^2(Q)$ . Then we conclude that the ground state of  $K_{\text{PF}}$  is unique if it exists.

This paper is organized as follows: Section 2 is devoted to constructing a strongly continuous symmetric semigroup  $(T_t)_{t \geq 0}$  and defining the self-adjoint operator  $K_{\text{PF}}$ . In Section 3 we show the spatial exponential decay of bound states of  $K_{\text{PF}}$  pointwise. Section 4 is an appendix.

## 2 Generalized Pauli-Fierz Hamiltonian

### 2.1 Definitions

Let us begin with defining a generalized Pauli-Fierz Hamiltonian by a path measure. We use the notation  $\mathbb{E}_P$  for the expectation with respect to a probability measure  $P$ , i.e.,  $\int \cdots dP = \mathbb{E}_P[\cdots]$ . Let  $\mathcal{S}_{\text{real}} = \mathcal{S}_{\text{real}}(\mathbb{R}^d)$  be the set of real-valued Schwartz test functions on  $\mathbb{R}^d$ . We set  $Q = \oplus_{j=1}^{d-1} \mathcal{S}_{\text{real}}$ . There exist a  $\sigma$ -field  $\mathcal{B}$ , a probability measure  $\mu$  on a measurable space  $(Q, \mathcal{B})$  and a Gaussian random variable  $\mathcal{A}(\Phi)$  indexed by  $\Phi = (\Phi_1, \dots, \Phi_{d-1}) \in \oplus_{j=1}^{d-1} L^2_{\text{real}}(\mathbb{R}^d)$  such that

$$\mathbb{E}_\mu[\mathcal{A}(\Phi)] = 0 \tag{2.1}$$

and the covariance is given by

$$\mathbb{E}_\mu[\mathcal{A}(\Phi)\mathcal{A}(\Psi)] = \frac{1}{2} \sum_{j=1}^{d-1} (\Phi_j, \Psi_j)_{L^2(\mathbb{R}^d)}. \tag{2.2}$$

Throughout the scalar product on Hilbert space  $\mathcal{L}$  is denoted by  $(F, G)_{\mathcal{L}}$ , where it is antilinear in  $F$  and linear in  $G$ . We omit  $\mathcal{L}$  when no confusion arises. For general  $\Phi \in \oplus_{j=1}^{d-1} L^2(\mathbb{R}^d)$ ,  $\mathcal{A}(\Phi)$  is defined by

$$\mathcal{A}(\Phi) = \mathcal{A}(\Re \Phi) + i\mathcal{A}(\Im \Phi). \tag{2.3}$$

Thus  $\mathcal{A}(\Phi)$  is linear in  $\Phi$  over  $\mathbb{C}$ . The Boson Fock space is defined by  $L^2(Q, d\mu) = L^2(Q)$ . It is known that the linear hull of

$$\{:\mathcal{A}(\phi_1) \cdots \mathcal{A}(\phi_n): |\phi_j \in \oplus^{d-1} L^2(\mathbb{R}^d), j = 1, \dots, n, n \geq 0\} \quad (2.4)$$

is dense in  $L^2(Q)$ , where  $:X:$  denotes the Wick product of  $X$ . See Section 4 for the definition of Wick product. Let us define the free field Hamiltonian  $H_f(m)$  on  $L^2(Q)$ . Define the map  $\Gamma(T) : L^2(Q) \rightarrow L^2(Q)$  by  $\Gamma(T)1 = 1$  and

$$\Gamma(T) : \mathcal{A}(\phi_1) \cdots \mathcal{A}(\phi_n) := \mathcal{A}(T\phi_1) \cdots \mathcal{A}(T\phi_n) : \quad (2.5)$$

for a contraction operator  $T$  on  $\oplus^{d-1} L^2(\mathbb{R}^d)$ . Then  $\Gamma(T)$  is also contraction on (2.4) and can be uniquely extended to the contraction operator on the whole space  $L^2(Q)$ , which is denoted by the same symbol  $\Gamma(T)$ . We can check that  $\Gamma(T)\Gamma(S) = \Gamma(TS)$ . Then  $\{\Gamma(e^{-ith})\}_{t \in \mathbb{R}}$  for a self-adjoint operator  $h$  defines the strongly continuous one-parameter unitary group on  $L^2(Q)$ . The self-adjoint generator of  $\{\Gamma(e^{-ith})\}_{t \in \mathbb{R}}$  is denoted by  $d\Gamma(h)$ , i.e.,

$$\Gamma(e^{-ith}) = e^{-itd\Gamma(h)}, \quad t \in \mathbb{R}. \quad (2.6)$$

Let

$$h = \oplus^{d-1} \omega(-i\partial), \quad (2.7)$$

where

$$\omega(k) = \sqrt{|k|^2 + m^2}, \quad m \geq 0, \quad k \in \mathbb{R}^d. \quad (2.8)$$

Then we set

$$H_f(m) = d\Gamma(h) \quad (2.9)$$

and it is called the free field Hamiltonian on  $L^2(Q)$ . Let  $p = -i\partial = (-i\partial_{x_1}, \dots, -i\partial_{x_d})$  be momentum operators in  $L^2(\mathbb{R}_x^d)$ . We define the Schrödinger operator  $H_p$  by

$$H_p = \frac{1}{2}p^2 + V, \quad (2.10)$$

where  $V$  denotes a real-valued external potential. The conditions on  $V$  will be required later. The zero coupling Hamiltonian is now given by the self-adjoint operator

$$H_p \otimes 1 + 1 \otimes H_f(m) \quad (2.11)$$

on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}_x^d) \otimes L^2(Q). \quad (2.12)$$

The Pauli-Fierz Hamiltonian  $H_{\text{PF}}$  is defined by replacing  $p \otimes 1$  in zero coupling Hamiltonian (2.11) with  $p \otimes 1 + \sqrt{\alpha}\mathcal{A}$ , where  $\alpha \geq 0$  is a coupling constant and

$$\mathcal{A}_\mu = \int_{\mathbb{R}^d}^\oplus \mathcal{A}_\mu(x) dx \quad (2.13)$$

is the so-called quantized radiation field. Here we used the identification  $\mathcal{H} \cong \int_{\mathbb{R}^d}^\oplus L^2(Q) dx$ . We shall define  $\mathcal{A}_\mu(x)$  below. Let

$$\rho_\mu^j(\cdot, x) = (\hat{\phi}_\mu^j \overline{\Psi(\cdot, x)} / \sqrt{\omega}), \quad j = 1, \dots, d-1, \quad \mu = 1, \dots, d, \quad (2.14)$$

where  $\phi_\mu^j$  is a cutoff function and  $\hat{X}$  (resp.  $\check{X}$ ) denotes the (resp. inverse) Fourier transform of  $X$ . Note that  $\hat{\rho}_\mu^j(k, x) = \hat{\phi}_\mu^j(k) \Psi(k, x) / \sqrt{\omega(k)}$ . Examples of cutoff functions are given later. The quantized radiation field is defined by

$$\mathcal{A}_\mu(x) = \mathcal{A} \left( \bigoplus_{j=1}^{d-1} \rho_\mu^j(x) \right), \quad \mu = 1, \dots, d, \quad (2.15)$$

for each  $x \in \mathbb{R}^d$ . Now we arrive at the definition of the Pauli-Fierz Hamiltonian. It is defined by

$$H_{\text{PF}} = \frac{1}{2}(p \otimes 1 + \sqrt{\alpha}\mathcal{A})^2 + V \otimes 1 + 1 \otimes H_{\text{f}}(m). \quad (2.16)$$

We omit  $\otimes$  for notational convenience in what follows. Then  $H_{\text{PF}}$  is expressed as

$$H_{\text{PF}} = \frac{1}{2}(p + \sqrt{\alpha}\mathcal{A})^2 + V + H_{\text{f}}(m). \quad (2.17)$$

**Assumption 2.1** Suppose that  $\hat{\rho}_\mu^j \in C_b^1(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$  and

$$\omega \hat{\rho}_\mu^j, \hat{\rho}_\mu^j, \hat{\rho}_\mu^j / \sqrt{\omega}, \partial_{x_\mu} \hat{\rho}_\mu^j, \partial_{x_\mu} \hat{\rho}_\mu^j / \sqrt{\omega} \in L^\infty(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d)). \quad (2.18)$$

Under Assumption 2.1 it follows that

$$\|(p \cdot \mathcal{A} + \mathcal{A} \cdot p)F\| \leq c_1 \|(p^2 + H_{\text{f}}(m) + 1)F\|, \quad (2.19)$$

$$\|\mathcal{A} \cdot \mathcal{A} F\| \leq c_2 \|(H_{\text{f}}(m) + 1)F\|. \quad (2.20)$$

Moreover  $H_{\text{PF}}$  is self-adjoint on  $D(p^2) \cap D(H_{\text{f}}(m))$  under Assumption 2.1. See [Hir00-b, Hir01, HH08] for the proof. We give examples of cutoff functions  $\rho_\mu^j$ .

**Example 2.2 (Standard Pauli-Fierz Hamiltonian)** The standard Pauli-Fierz Hamiltonian is defined by  $H_{\text{PF}}$  with the dimension  $d = 3$ ,  $m = 0$ , and

$$\Psi(k, x) = e^{+ikx}, \quad \hat{\phi}_\mu^j(k) = \hat{\varphi}(k)e_\mu(k, j)/\sqrt{\omega},$$

where  $e(k, j) = (e_1(k, j), e_2(k, j), e_3(k, j))$ ,  $j = 1, 2$ , denote polarization vectors, and  $\hat{\varphi}$  is an ultraviolet cutoff function. Suppose that  $\sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^d)$ . Then  $\rho_\mu^j(k, x) \in C_b^1(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$  and (2.18) is fulfilled.

**Example 2.3 (The Pauli-Fierz Hamiltonian with a variable mass)** The Pauli-Fierz Hamiltonian with a variable mass  $v$  instead of  $m$  is studied in [Hid10]. Then  $d = 3$ ,  $m = 0$ , and  $\Psi(k, x)$  is the unique solution to the Lippman-Schwinger equation [Ike60]:

$$\Psi(k, x) = e^{+ikx} - \frac{1}{4\pi} \int \frac{e^{i|k||x-y|}v(y)}{|x-y|} \Psi(k, y) dy. \quad (2.21)$$

$\Psi(k, x)$  formally satisfies

$$(-\Delta_x + v(x))\Psi(k, x) = |k|^2\Psi(k, x), \quad k \neq 0.$$

It is established that the Pauli-Fierz Hamiltonian with a variable mass has a ground state for arbitrary values of coupling constants when  $|v(x)| \leq C(1+|x|^2)^{-\beta/2}$ ,  $\beta > 3$ , with some constant  $C$ . Then it is also seen that

$$|\Psi(k, x) - e^{ikx}| \leq C(1+|x|^2)^{-1/2}. \quad (2.22)$$

Since

$$\partial_{x_\mu} \Psi(k, x) = ik_\mu e^{ikx} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \left( \frac{1}{|x-y|} - i|k| \right) \frac{(x_\mu - y_\mu) e^{i|k||x-y|}v(y)}{|x-y|^2} \Psi(k, y) dy, \quad (2.23)$$

it follows that

$$\sup_{k \in D, x \in \mathbb{R}_x^d} |\partial_{x_\mu} \Psi(k, x)| < \infty \quad (2.24)$$

for any compact set  $D$  but  $D \not\ni 0$ . Let  $\text{supp} \hat{\phi}_\mu^j \subset D$ . Then  $\rho_\mu^j \in C_b^1(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$  follows from (2.22) and (2.24). In addition to condition  $\text{supp} \hat{\phi}_\mu^j \subset D$  let us suppose that  $\hat{\phi}_\mu^j/\sqrt{\omega}, \sqrt{\omega}\hat{\phi}_\mu^j, \hat{\phi}_\mu^j/\omega \in L^2(\mathbb{R}_k^d)$ , then (2.18) is fulfilled.



## 2.2 Feynman-Kac type formulae

Let us prepare the Euclidean version of the quantized radiation field  $\mathcal{A}(\Phi)$  to construct a functional integral representation of  $e^{-tH_{\text{PF}}}$  in the same way as [Hir97]. Let  $Q_E = \oplus^{d-1} \mathcal{S}_{\text{real}}(\mathbb{R}^{d+1})$ . There exist a probability measure  $\mu_E$  on a measurable space  $(Q_E, \mathcal{B}_E)$  and a Gaussian random variable  $\mathcal{A}_E(\Phi)$  indexed by  $\Phi \in \oplus^{d-1} L^2(\mathbb{R}^{d+1})$  such that

$$\mathbb{E}_{\mu_E}[\mathcal{A}_E(\Phi)] = 0$$

and the covariance is given by

$$\mathbb{E}_{\mu_E}[\mathcal{A}_E(\Phi)\mathcal{A}_E(\Psi)] = \frac{1}{2} \sum_{j=1}^{d-1} (\Phi_j, \Psi_j)_{L^2(\mathbb{R}^{d+1})}.$$

Both  $L^2(Q)$  and  $L^2(Q_E)$  are connected through the second quantization of the family of isometry  $\{\mathbf{j}_t\}_{t \in \mathbb{R}}$  between  $L^2(\mathbb{R}^d)$  and  $L^2(\mathbb{R}^{d+1})$ :

$$\widehat{\mathbf{j}_t f}(k_0, k) = \frac{e^{-ik_0 t}}{\sqrt{\pi}} \sqrt{\omega(k)/(\omega(k)^2 + |k_0|^2)} \hat{f}(k). \quad (2.25)$$

Define  $\mathbf{J}_t = \Gamma(\oplus^{d-1} \mathbf{j}_t) : L^2(Q) \rightarrow L^2(Q_E)$ . From the identity  $\mathbf{j}_t^* \mathbf{j}_s = e^{-|t-s|\omega(-i\partial)}$  it follows that  $\mathbf{J}_t^* \mathbf{J}_s = e^{-|t-s|H_{\text{f}}(m)}$ .

Set  $\mathcal{X} = C([0, \infty); \mathbb{R}^d)$  be the set of continuous paths on  $[0, \infty)$ . Let  $(B_t)_{t \geq 0}$  denote the  $d$ -dimensional Brownian motion starting at  $x \in \mathbb{R}^d$  on  $(\mathcal{X}, \mathcal{B}(\mathcal{X}), P^x)$  with the Wiener measure  $P^x$ . I.e.,  $P^x(B_0 = x) = 1$ . Let  $C_{\text{b}}^n(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$  be the set of strongly  $n$ -times differentiable  $L^2(\mathbb{R}^d)$ -valued functions on  $\mathbb{R}^d$  such that  $\sup_x \|\partial_x^z f(x)\|_{L^2(\mathbb{R}^d)} < \infty$  for  $|z| \leq n$ . For  $f_\mu \in C_{\text{b}}^1(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$ ,  $\mu = 1, \dots, d$ , we can define an  $L^2(\mathbb{R}^d)$ -valued Stratonovich integral:

$$\sum_{\mu=1}^d \int_0^t f_\mu(B_s) \circ dB_s^\mu = \int_0^t f(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \partial \cdot f(B_s) ds, \quad (2.26)$$

where  $f(B_s) \cdot dB_s = \sum_{\mu=1}^d f_\mu(B_s) dB_s^\mu$  and  $\partial \cdot f(B_s) = \sum_{\mu=1}^d (\partial_{x_\mu} f_\mu)(B_s)$ . We also define an  $L^2(\mathbb{R}^{d+1})$ -valued Stratonovich integral by

$$\sum_{\mu=1}^d \int_0^t \mathbf{j}_s f_\mu(B_s) \circ dB_s^\mu = \sum_{\mu=1}^d \lim_{n \rightarrow \infty} \int_{t(j-1)/n}^{tj/n} \mathbf{j}_{t(j-1)/n} f_\mu(B_s) \circ dB_s^\mu, \quad (2.27)$$

where  $\lim_{n \rightarrow \infty}$  is a strong limit in  $L^2(\mathcal{X}; L^2(\mathbb{R}^{d+1}))$ . By the Itô isometry we have the identity for  $S \leq T$

$$\begin{aligned} \mathbb{E}^x \left[ \left( \int_0^T j_s f(B_s) \cdot dB_s, \int_0^S j_s g(B_s) \cdot dB_s \right)_{L^2(\mathbb{R}^{d+1})} \right] \\ = \sum_{\mu=1}^d \int_0^S \mathbb{E}^x [(f_\mu(B_s), g_\mu(B_s))] ds \end{aligned}$$

Hence we have the bound

$$\mathbb{E}^x \left[ \left\| \sum_{\mu=1}^d \int j_s f_\mu(B_s) \circ dB_s^\mu \right\|^2 \right] \leq \int_0^t ds \mathbb{E}^x \left[ 2 \sum_{\mu=1}^d \|f_\mu(B_s)\|^2 + \frac{1}{2} \|\partial \cdot f(B_s)\|^2 \right] \quad (2.28)$$

The next proposition is fundamental.

**Proposition 2.4** *Let  $V$  be bounded. Suppose Assumption 2.1. Then*

$$(F, e^{-tH_{\text{PF}}} G) = \int dx \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} \left( J_0 F(B_0), e^{i\sqrt{\alpha} \mathcal{A}_E(K_t)} J_t G(B_t) \right)_{L^2(Q)} \right], \quad (2.29)$$

$K_t$  is the  $\oplus^{d-1} L^2(\mathbb{R}^{d+1})$ -valued stochastic integral given by

$$K_t = \oplus_{j=1}^{d-1} \sum_{\mu=1}^d \int_0^t j_s \rho_\mu^j(\cdot, B_s) \circ dB_s^\mu. \quad (2.30)$$

Here

$$\sum_{\mu=1}^d \int_0^t j_s \rho_\mu^j(\cdot, B_s) \circ dB_s^\mu = \int_0^t j_s \rho^j(\cdot, B_s) \cdot dB_s + \frac{1}{2} \int_0^t j_s \partial \cdot \rho^j(\cdot, B_s) ds.$$

*Proof:* Suppose that  $\hat{\rho}_\mu^j \in C_b^2(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$ . Then (2.29) is proven in the same way as [Hir00-b, Lemma 4.8]. Next we suppose that  $\hat{\rho}_\mu^j(k, x) \in C_b^1(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$ . Let

$\chi \in C^\infty(\mathbb{R}^d)$  and  $\varphi \in C_0^\infty(\mathbb{R}^d)$  be such that  $\chi(x) = \begin{cases} 1, & |x| < 1, \\ < 1, & 1 \leq |x| \leq 2, \\ 0, & 2 < |x|, \end{cases} \quad \varphi \geq 0$

and  $\int \varphi(x) dx = 1$ . Define  $\chi_N(x) = \chi(x/N)$  and  $\varphi_n(x) = \varphi(x/n) n^{-d/2}$ . Let

$$\begin{aligned} \hat{\rho}_\mu^j(k, x)_{M,n} &= (\varphi_n * (\rho_\mu^j(k, \cdot) \chi_N(\cdot))) (x), \\ \hat{\rho}_\mu^j(k, x)_M &= \rho_\mu^j(k, x) \chi_M(x). \end{aligned}$$

We note that  $\hat{\rho}_\mu^j(k, x)_{M,n} \in C_b^\infty(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$ . Since  $\hat{\rho}_\mu^j(k, x)_{M,n} \rightarrow \hat{\rho}_\mu^j(k, x)_M$  in  $L^p(\mathbb{R}_x^d, L^2(\mathbb{R}_k^d))$  for  $1 \leq p < \infty$  as  $n \rightarrow \infty$ , there exists a subsequence  $n'$  such that  $\hat{\rho}_\mu^j(k, x)_{M,n'} \rightarrow \hat{\rho}_\mu^j(k, x)_M$  strongly in  $L^2(\mathbb{R}_k^d)$  for almost everywhere  $x \in \mathbb{R}^d$ . Furthermore  $\hat{\rho}_\mu^j(k, x)_M \rightarrow \hat{\rho}_\mu^j(k, x)$  for each  $x \in \mathbb{R}^d$  in  $L^2(\mathbb{R}_k^d)$ . Then

$$\lim_{M \rightarrow \infty} \lim_{n' \rightarrow \infty} \hat{\rho}_\mu^j(k, x)_{M,n} = \hat{\rho}_\mu^j(k, x) \quad (2.31)$$

strongly in  $L^2(\mathbb{R}_k^d)$  for almost everywhere  $x \in \mathbb{R}^d$ . In the same way as above we can also see that

$$\lim_{M \rightarrow \infty} \lim_{n' \rightarrow \infty} \partial_x^z \hat{\rho}_\mu^j(k, x)_{M,n} = \partial_x^z \hat{\rho}_\mu^j(k, x) \quad (2.32)$$

strongly in  $L^2(\mathbb{R}_k^d)$  for almost everywhere  $x \in \mathbb{R}^d$  for  $|z| \leq 1$ . Thus (2.29) holds with  $\hat{\rho}_\mu^j$  replaced by  $\hat{\rho}_\mu^j(k, x)_{M,n'}$ .  $H_{\text{PF}}$  with  $\rho_\mu^j$  replaced by  $\hat{\rho}_\mu^j(k, x)_{M,n'}$  is denoted by  $H_{\text{PF}}(M, n')$ . Let  $F \in C_0^\infty \otimes D(H_f(m))$ . Then we can prove directly that

$$\lim_{M \rightarrow \infty} \lim_{n' \rightarrow \infty} H_{\text{PF}}(M, n')F = H_{\text{PF}}F.$$

Since  $C_0^\infty \otimes D(H_f(m))$  is a core of  $H_{\text{PF}}(M, n')$  and  $H_{\text{PF}}$ ,

$$\lim_{M \rightarrow \infty} \lim_{n' \rightarrow \infty} e^{-tH_{\text{PF}}(M, n')} = e^{-tH_{\text{PF}}} \quad (2.33)$$

strongly. Moreover

$$(F, e^{-tH_{\text{PF}}(M, n')}G) = \int dx \mathbb{E}^x \left[ \left( J_0 F(x), e^{i\sqrt{\alpha}\mathcal{A}_E(K_t(M, n'))} e^{-\int_0^t V(B_s)} J_t G(B_t) \right) \right], \quad (2.34)$$

where  $K_t(M, n')$  is defined by  $K_t$  with  $\rho_\mu^j(k, x)$  replaced by  $\hat{\rho}_\mu^j(k, x)_{M,n'}$ . Operator  $N = d\Gamma(1)$  is called the number operator in  $L^2(Q)$ . Let  $F \in D(N)$ . Then the bound

$$\|\mathcal{A}(\Phi)F\| \leq 2\|\Phi\|(N+1)^{1/2}F\|$$

is known. From (2.34) and

$$|e^{i\sqrt{\alpha}\mathcal{A}_E(K_t(M, n'))} - e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)}| \leq |\mathcal{A}_E(K_t(M, n') - K_t)|$$

it follows that

$$\begin{aligned}
& |(F, e^{-tH_{\text{PF}}(M, n')}G) - (F, e^{-tH_{\text{PF}}}G)| \\
& \leq \sqrt{\alpha} \int dx \mathbb{E}^x \left[ \left( |J_0 F(x)|, |\mathcal{A}_E(K_t(M, n') - K_t)| e^{-\int_0^t V(B_s)} |J_t G(B_t)| \right) \right] \\
& \leq C\sqrt{\alpha} \int dx \|(N+1)^{1/2} F(x)\| \mathbb{E}^x \left[ \|K_t(M, n') - K_t\| \|G(B_t)\| \right] \\
& \leq C\sqrt{\alpha} \int dx \|(N+1)^{1/2} F(x)\| \left( \mathbb{E}^x [\|K_t(M, n') - K_t\|^2] \right)^{1/2} \left( \mathbb{E}^x [\|G(B_t)\|^2] \right)^{1/2}.
\end{aligned}$$

We estimate  $\mathbb{E}^x [\|K_t(M, n') - K_t\|^2]$ . By (2.28) we have

$$\mathbb{E}^x [\|K_t(M, n') - K_t\|^2] \leq \sum_{j=1}^{d-1} \int_0^t \mathbb{E}^x \left[ 2 \sum_{\mu=1}^d \|\delta \rho_\mu^j(B_s)\|^2 + \frac{1}{2} \|\delta \partial \cdot \rho^j(B_s)\|^2 \right] ds.$$

where  $\delta f = f - f_{M, n'}$ . By (2.31) and (2.32) we see that

$$\lim_{M \rightarrow \infty} \lim_{n' \rightarrow \infty} \mathbb{E}^x [\|K_t(M, n') - K_t\|^2] = 0$$

for each  $x \in \mathbb{R}^d$ . Then by the Lebesgue dominated convergence theorem we have

$$\lim_{M \rightarrow \infty} \lim_{n' \rightarrow \infty} \text{r.h.s. (2.34)} = \int dx \mathbb{E}^x \left[ \left( J_0 F(x), e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} e^{-\int_0^t V(B_s)} J_t G(B_t) \right) \right]. \quad (2.35)$$

Then (2.29) also holds for  $\rho_\mu^j \in C_b^1(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$ . Thus the proposition follows. QED

### 2.3 One-parameter symmetric semigroup and generalized Pauli-Fierz Hamiltonian

We can extend functional integral representations in Proposition 2.4 to more general external potentials and  $\rho_\mu^j$ .

**Definition 2.5 (Kato class potentials)** *External potential  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  is called a Kato-class potential if and only if*

$$\begin{cases} \sup_{x \in \mathbb{R}^d} \int_{B_r(x)} |\lambda(x-y)V(y)| dy < \infty & d = 1, \\ \lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{B_r(x)} |\lambda(x-y)V(y)| dy = 0 & d \geq 2 \end{cases} \quad (2.36)$$

holds, where  $B_r(x)$  denotes the closed ball of radius  $r$  centered at  $x$ , and

$$\lambda(x) = \begin{cases} 1, & d = 1, \\ -\log|x|, & d = 2, \\ |x|^{2-d}, & d \geq 3. \end{cases} \quad (2.37)$$

We denote the set of Kato-class potential by  $\mathcal{K}_{kato}$ .

An equivalent characterization of Kato-class is as follows:

**Proposition 2.6** *A function  $V$  is in  $\mathcal{K}_{kato}$  if and only if*

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[ \int_0^t |V(B_s)| ds \right] = 0. \quad (2.38)$$

*Proof:* See e.g., [AS82, CFKS87, Sim82].

QED

**Definition 2.7** *Let  $\mathcal{K}$  be the set of external potential  $V = V_+ - V_-$  such that  $0 \leq V_+ \in L^1_{loc}(\mathbb{R}^d)$  and  $0 \leq V_- \in \mathcal{K}_{kato}$ .*

**Example 2.8** *In [AS82, Sim82], it is shown that  $L^p_u(\mathbb{R}^d) \subset \mathcal{K}_{kato}$  where*

$$L^p_u(\mathbb{R}^d) = \left\{ f \left| \sup_x \int_{|x-y| \leq 1} |f(x)|^p dx < \infty \right. \right\}$$

with

$$p \begin{cases} = 1, & d = 1, \\ > d/2, & d \geq 2. \end{cases} \quad (2.39)$$

*In particular let  $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$  with (2.39), then  $V \in \mathcal{K}_{kato}$ .*

**Example 2.9** *Let  $d = 3$  and  $V(x) = P(x) - \frac{a}{|x|^b}$ , where  $a \geq 0$ ,  $0 \leq b < 2$  and  $P(x) = \sum_{j=0}^{2n} a_j x^j$  is a polynomial such that  $a_{2n} > 0$ . Then  $V \in \mathcal{K}$ .*

Now we shall see that the random variable  $\int_0^t V_\pm(B_s) ds$  is integrable with respect to the Wiener measure  $P^x$  for  $V \in \mathcal{K}$ .

**Lemma 2.10** *Let  $0 \leq V \in L^1_{loc}(\mathbb{R}^d)$ . Then  $P^x \left( \int_0^t V(B_s) ds < \infty \right) = 1$  for each  $x \in \mathbb{R}^d$ .*

*Proof:* Since  $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ , we can see that  $\mathbb{E}^x[\int_0^t 1_N V(B_s) ds] < \infty$  for the indicator function  $1_N(k) = \begin{cases} 1, & |k| \leq N, \\ 0, & |k| > N, \end{cases}$ . Then there exists a measurable set  $\mathcal{N}_N \subset \mathcal{X}$  such that  $P^x(\mathcal{N}_N) = 0$  and  $\int_0^t 1_N(B_s) V(B_s) ds < \infty$  for  $\omega \in \mathcal{X} \setminus \mathcal{N}_N$ . Set  $\mathcal{N} = \cup_{N=1}^{\infty} \mathcal{N}_N$ . For  $\omega \in \mathcal{X} \setminus \mathcal{N}$  we can see that  $\int_0^t 1_N(B_s(\omega)) V(B_s(\omega)) ds < \infty$  for arbitrary  $N \geq 1$ . Let  $\omega \in \mathcal{X} \setminus \mathcal{N}$ . There exists  $N = N(\omega) \geq 1$  such that  $\sup_{0 \leq s \leq t} |B_s(\omega)| < N$ . Henceforce

$$\int_0^t V(B_s(\omega)) ds = \int_0^t 1_N(B_s(\omega)) V(B_s(\omega)) ds < \infty, \quad \omega \in \mathcal{X} \setminus \mathcal{N}.$$

Thus the lemma follows. QED

When  $V_- \in \mathcal{K}_{kato}$ , it can be seen that the exponent  $e^{\int_0^t V(B_s) ds}$  is integrable with respect to  $P^x$ , and the supremum of  $\mathbb{E}^x \left[ e^{\int_0^t V(B_s) ds} \right]$  in  $x$  is finite. We shall check it.

**Lemma 2.11** *Let  $V \in \mathcal{K}_{kato}$ . Then there exists  $\beta > 0$  and  $\gamma > 0$  such that*

$$\sup_x \mathbb{E}^x \left[ e^{\int_0^t V(B_s) ds} \right] < \gamma e^{\beta t} \quad (2.40)$$

*Furthermore when  $V \in L^p(\mathbb{R}^d)$  with  $p \begin{cases} = 1, & d = 1, \\ > d/2, & d \geq 2, \end{cases}$  there exists  $C$  such that*

$$\beta \leq C \|V\|_p. \quad (2.41)$$

*Proof:* By Proposition 2.6 there exists  $t^* > 0$  such that

$$\alpha_t = \sup_x \mathbb{E}^x \left[ \int_0^t V(B_s) ds \right] < 1$$

for all  $t \leq t^*$ , and  $\alpha_t \rightarrow 0$  as  $t \rightarrow 0$ . It is known as Khasminskii's lemma that

$$\sup_x \mathbb{E}^x \left[ e^{\int_0^t V(B_s) ds} \right] < \frac{1}{1 - \alpha_t} \quad (2.42)$$

for all  $t \leq t^*$ . By means of the Markov property of the Brownian motion we have

$$\mathbb{E}^x \left[ e^{-\int_0^{2t^*} V(B_s) ds} \right] = \mathbb{E}^x \left[ e^{-\int_0^{t^*} V(B_s) ds} \mathbb{E}^{B_{t^*}} \left[ e^{-\int_0^{t^*} V(B_s) ds} \right] \right] \leq \left( \frac{1}{1 - \alpha_{t^*}} \right)^2.$$

Repeating this procedure we can see that

$$\sup_x \mathbb{E}^x \left[ e^{\int_0^t V_-(B_s) ds} \right] \leq \left( \frac{1}{1 - \alpha_{t^*}} \right)^{[t/t^*]+1} \quad (2.43)$$

for all  $t > 0$ , where  $[z] = \max\{w \in \mathbb{Z} | w \leq z\}$ . Set  $\gamma = \left(\frac{1}{1-\alpha_t^*}\right)$  and  $\beta = \log\left(\frac{1}{1-\alpha_t^*}\right)^{1/t^*}$ . Then (2.40) is proven. Next we prove (2.41). Suppose  $V \in L^p(\mathbb{R}^d)$ . In the case of  $d = 1$  we directly see that

$$\alpha_t = \int_0^t \mathbb{E}^x [V(B_s)] ds \leq \int_0^t (2\pi s)^{-1/2} ds \|V\|_1. \quad (2.44)$$

Next we let  $d \geq 2$  and  $q$  be such that  $\frac{1}{p} + \frac{1}{q} = 1$ . The following estimates are due to [AS82, proof of Theorem 4.5]. Let an arbitrary  $\epsilon > 0$  be fixed. We have

$$\begin{aligned} & \int_0^t \mathbb{E}^x [|V(B_s)|] ds \\ &= \int_0^t \mathbb{E}^x [|V(B_s)| \chi_{|B_s-x| \geq \epsilon}] ds + \int_0^t \mathbb{E}^x [|V(B_s)| \chi_{|B_s-x| < \epsilon}] ds \\ &\leq t \int_{|y| \geq \epsilon} (2\pi t)^{-d/2} e^{-|y|^2/(2t)} |V(x+y)| dy + e^t \int_0^\infty \mathbb{E}^x [e^{-s} |V(B_s)| \chi_{|B_s-x| < \epsilon}] ds. \end{aligned}$$

It is easy to see that

$$t \int_{|y| \geq \epsilon} (2\pi t)^{-d/2} e^{-|y|^2/(2t)} |V(x+y)| dy \leq t(2\pi)^{-d/2} \left( \int e^{-q|y|^2/2} dy \right)^{1/q} \|V\|_p. \quad (2.45)$$

Let  $f$  be the integral kernel of  $(\frac{1}{2}p^2 + 1)^{-1}$ . Then we see that

$$\int_0^\infty ds \mathbb{E}^x [e^{-s} |V(B_s)| \chi_{|B_s-x| < \epsilon}] \leq \int_{|x-y| < \epsilon} f(x-y) |V(y)| dy.$$

Since  $|f(z)| \leq C\lambda(z)$  for  $|z| \leq \frac{1}{2}$  with some constant  $C$ , we have

$$\int_0^\infty ds \mathbb{E}^x [e^{-s} |V(B_s)| \chi_{|B_s-x| < \epsilon}] \leq C \int_{|x-y| < \epsilon} \lambda(x-y) |V(y)| dy$$

and then

$$\int_0^\infty ds \mathbb{E}^x [e^{-s} |V(B_s)| \chi_{|B_s-x| < \epsilon}] \leq C \left( \int_{|z| < \epsilon} \lambda(z)^q dy \right)^{1/q} \|V\|_p \quad (2.46)$$

by the Hölder inequality. Hence from (2.44), (2.45) and (2.46), there exists  $C_t(\epsilon)$  such that  $\alpha_t \leq C_t(\epsilon) \|V\|_p$  and  $\lim_{t \rightarrow 0} C_t(\epsilon) = C \left( \int_{|z| < \epsilon} \lambda(z)^q dy \right)^{1/q}$ . Then for sufficiently small  $T$  and  $\epsilon$  we have  $\beta \leq \left( \frac{1}{1-C_T(\epsilon) \|V\|_p} \right)^{1/T}$  and then there exists  $D_T$  such that  $\beta \leq D_T \|V\|_p$ . Then (2.41) follows. QED

The functional integral representation (2.29) introduced in Proposition 2.4 is well defined not only for bounded external potentials and  $\rho_\mu^j$  satisfying (2.18) but also more general external potentials and  $\rho_\mu^j$ . We can identify Hilbert space  $\mathcal{H}$  with  $L^2(\mathbb{R}^d \times Q)$  with the scalar product  $(F, G) = \int dx (F(x), G(x))_{L^2(Q)}$ . The functional integral representation of  $(F, e^{-tH_{\text{PF}}} G)$  is also given by

$$(F, e^{-tH_{\text{PF}}} G) = \int dx \left( F(x), \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} J_0^* e^{i\sqrt{\alpha} \mathcal{A}_E(K_t)} J_t G(B_t) \right] \right)_{L^2(Q)}.$$

From this expression we shall define  $(T_t)_{t \geq 0}$  by (2.47) below.

**Assumption 2.12** *We suppose that  $V \in \mathcal{K}$  and  $\hat{\rho}_\mu^j = \hat{\rho}_\mu^j(k, x) \in C_b^1(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$ .*

Note that under Assumption 2.12,  $\mathcal{A}_\mu(x)$  is *not* relatively bounded with respect to  $H_f(m)$  in the case of  $m = 0$ . Under Assumption 2.12 however we define the family of linear operators  $\{T_t\}_{t \geq 0}$  on  $\mathcal{H}$  by

$$T_t F(x) = \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} J_0^* e^{i\sqrt{\alpha} \mathcal{A}_E(K_t)} J_t F(B_t) \right] \quad (2.47)$$

for all  $t \geq 0$ . Note that  $K_t$  is well defined since  $\hat{\rho}_\mu^j \in C_b^1(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$ .

**Lemma 2.13** *Suppose Assumption 2.12. Then  $T_t$  is bounded on  $\mathcal{H}$  for  $t \geq 0$ .*

*Proof:* By the definition of  $T_t$  we have

$$\|T_t F\|_{\mathcal{H}}^2 \leq \int dx \mathbb{E}^x \left[ e^{-2 \int_0^t V(B_s) ds} \right] \mathbb{E}^x \left[ \|F(B_t)\|_{L^2(Q)}^2 \right].$$

Since  $V \in \mathcal{K}$ ,  $C = \sup_x \mathbb{E}^x \left[ e^{-2 \int_0^t V(B_s) ds} \right] < \infty$ . Thus  $\|T_t F\|_{\mathcal{H}}^2 \leq C \|F\|_{\mathcal{H}}^2$  follows. QED

In what follows we shall show that  $\{T_t\}_{t \geq 0}$  is a strongly continuous one-parameter symmetric semigroup on  $\mathcal{H}$ . In order to show it we introduce the second quantization of Euclidean group  $\{u_t, r\}$  on  $L^2(\mathbb{R}^{d+1})$ , where the time shift operator  $u_t : L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1})$  is defined by

$$u_t f(x_0, \mathbf{x}) = f(x_0 - t, \mathbf{x})$$

and the time reflection  $r : L^2(\mathbb{R}^{d+1}) \rightarrow L^2(\mathbb{R}^{d+1})$  by

$$r f(x_0, \mathbf{x}) = f(-x_0, \mathbf{x})$$



for  $x = (x_0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ . The second quantization of  $u_t$  and  $r$  are denoted by  $U_t : L^2(Q_E) \rightarrow L^2(Q_E)$  and  $R : L^2(Q_E) \rightarrow L^2(Q_E)$ , respectively. Note that  $r^* = r$ ,  $rr = r^*r = 1$ ,  $u_t^* = u_{-t}$  and  $u_t^*u_t = 1$  and that  $U_t$  and  $R$  are unitary. The time shift  $u_t$ , the time reflection  $r$  and isometry  $j_t : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^{d+1})$  satisfy the lemma below.

**Lemma 2.14** (1)  $u_t j_s = j_{s+t}$  and  $U_t J_s = J_{s+t}$ . (2)  $r j_s = j_{-s} r$  and  $RU_s = U_{-s} R$ .

*Proof:* By the definition of  $j_s$  we have

$$j_s f(x) = \frac{1}{\sqrt{\pi}(2\pi)^{(d+1)/2}} \int e^{i(k_0(x_0-s)+k \cdot \mathbf{x})} \frac{\sqrt{\omega(k)}}{\sqrt{\omega(k)^2 + |k_0|^2}} \hat{f}(k) dk_0 dk.$$

Then  $u_t j_s = j_{s+t}$  follows, and  $U_t J_s = \Gamma(u_t) \Gamma(j_s) = \Gamma(u_t j_s) = \Gamma(j_{s+t}) = J_{s+t}$ . (2) is similarly proven. QED

**Lemma 2.15** Suppose Assumption 2.12. Then it follows that  $T_t T_s = T_{t+s}$  for all  $t, s \geq 0$ .

*Proof:* By the definition of  $T_t$  we have

$$T_s T_t F(x) = \mathbb{E}^x \left[ e^{-\int_0^s V(B_r) dr} J_0^* e^{i\sqrt{\alpha} \mathcal{A}_E(K_s)} J_s \mathbb{E}^{B_s} \left[ e^{-\int_0^t V(B_r) dr} J_0^* e^{i\sqrt{\alpha} \mathcal{A}_E(K_t)} J_t F(B_t) \right] \right]. \quad (2.48)$$

Let  $E_s = J_s J_s^*$ ,  $s \in \mathbb{R}$ , be the family of projections. By the formulae  $J_s J_0^* = J_s J_s^* U_{-s}^* = E_s U_{-s}^*$  and  $J_t = U_{-s} J_{t+s}$ , (2.48) is expressed as

$$\begin{aligned} & T_s T_t F(x) \\ &= \mathbb{E}^x \left[ e^{-\int_0^s V(B_r) dr} J_0^* e^{i\sqrt{\alpha} \mathcal{A}_E(K_s)} E_s \mathbb{E}^{B_s} \left[ e^{-\int_0^t V(B_r) dr} U_{-s}^* e^{i\sqrt{\alpha} \mathcal{A}_E(K_t)} U_{-s} J_{t+s} F(B_t) \right] \right]. \end{aligned}$$

Since  $U_s$  is unitary we have

$$U_{-s}^* e^{i\sqrt{\alpha} \mathcal{A}_E(K_t)} U_{-s} = e^{i\sqrt{\alpha} \mathcal{A}_E(u_{-s}^* K_t)} \quad (2.49)$$

as an operator, where the exponent is given by

$$u_{-s}^* K_t = \oplus_{j=1}^{d-1} \sum_{\mu=1}^d \int_0^t j_{r+s} \rho_{\mu}^j(B_r) \circ dB_r^{\mu}.$$

Let  $(\mathcal{F}_t)_{t \geq 0}$  be the natural filtration of the Brownian motion  $(B_t)_{t \geq 0}$ . By the Markov property of the projections  $E_t$ 's [Sim74], we can neglect  $E_s$  in (2.49) and we have

$$\begin{aligned} & T_s T_t F(x) \\ &= \mathbb{E}^x \left[ e^{-\int_0^s V(B_r) dr} J_0^* e^{i\sqrt{\alpha} \mathcal{A}_E(K_s)} \mathbb{E}^x \left[ e^{-\int_s^{s+t} V(B_r) dr} e^{i\sqrt{\alpha} \mathcal{A}_E(K_s^{s+t})} J_{t+s} F(B_{s+t}) \middle| \mathcal{F}_s \right] \right], \end{aligned}$$

where  $\mathbb{E}^x[\cdots | \mathcal{F}_s]$  denotes the conditional expectation with respect to  $(\mathcal{F}_t)_{t \geq 0}$  and

$$K_s^{s+t} = \oplus_{j=1}^{d-1} \sum_{\mu=1}^d \int_s^{s+t} j_r \rho_\mu^j(B_r) \circ dB_r^\mu.$$

Hence we obtain that

$$T_s T_t F(x) = \mathbb{E}^x \left[ e^{-\int_0^{s+t} V(B_r) dr} J_0^* e^{i\sqrt{\alpha} \mathcal{A}_E(K_{s+t})} J_{s+t} F(B_{s+t}) \right] = T_{s+t} F(x)$$

and the lemma is proven. QED

Next we check the symmetric property of  $T_t$ .

**Lemma 2.16** *Suppose Assumption 2.12. Then it follows that  $T_t^* = T_t$  for  $t \geq 0$ .*

*Proof:* By the functional integral representation and the unitarity of the time-reflection  $R$  on  $L^2(Q_E)$ , we have

$$\begin{aligned} (F, T_t G) &= \int dx \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} \left( R J_0 F(B_0), R e^{i\sqrt{\alpha} \mathcal{A}_E(K_t)} R R J_t G(B_t) \right) \right] \\ &= \int dx \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} \left( J_0 F(B_0), e^{i\sqrt{\alpha} \mathcal{A}_E(r K_t)} J_{-t} G(B_t) \right) \right], \end{aligned}$$

where the exponent is  $r K_t = \oplus_{j=1}^{d-1} \sum_{\mu=1}^d \int_0^t j_{-s} \rho_\mu^j(B_s) \circ dB_s^\mu$ . By means of the time-shift  $U_t$  we also have

$$\begin{aligned} (F, T_t G) &= \int dx \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} \left( U_t J_0 F(B_0), U_t e^{i\sqrt{\alpha} \mathcal{A}_E(r K_t)} U_t^* U_t J_{-t} G(B_t) \right) \right] \\ &= \int dx \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} \left( J_t F(B_0), e^{i\sqrt{\alpha} \mathcal{A}_E(u_t r K_t)} J_0 G(B_t) \right) \right], \end{aligned}$$

where  $u_t r K_t = \oplus_{j=1}^{d-1} \sum_{\mu=1}^d \int_0^t j_{t-s} \rho_\mu^j(B_s) \circ dB_s^\mu$ . Finally we set  $\tilde{B}_s = B_{t-s} - B_t$ , which equals to  $B_s$  in law. Then we have

$$(F, T_t G) = \int dx \mathbb{E}^0 \left[ e^{-\int_0^t V(x + \tilde{B}_s) ds} \left( J_t F(x), e^{i\sqrt{\alpha} \mathcal{A}_E(\widetilde{u_t r K_t})} J_0 G(x + \tilde{B}_t) \right) \right], \quad (2.50)$$

where

$$\widetilde{u_t r K_t} = \oplus_{j=1}^{d-1} \sum_{\mu=1}^d \int_0^t \mathbf{j}_{t-s} \rho_\mu^j(x + \tilde{B}_s) \circ d\tilde{B}_s^\mu = \lim_{n \rightarrow \infty} \oplus_{j=1}^{d-1} \sum_{i=1}^n \Delta_j(i)$$

and  $\lim_{n \rightarrow \infty}$  is in the strong sense of  $L^2(\mathcal{X}; L^2(\mathbb{R}^{d+1}))$  and

$$\Delta_j(i) = \sum_{\mu=1}^d \int_{t(i-1)/n}^{ti/n} \mathbf{j}_{t-t(i-1)/n} \rho_\mu^j(x + \tilde{B}_s) \circ d\tilde{B}_s^\mu.$$

Then exchanging  $\int dx$  and  $\mathbb{E}^0$  in (2.50) we have

$$\begin{aligned} & (F, T_t G) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^0 \left[ \int dx e^{-\int_0^t V(x + \tilde{B}_s) ds} \left( \mathbf{J}_t F(x), e^{i\sqrt{\alpha} \mathcal{A}_E(\oplus_{j=1}^{d-1} \sum_{i=1}^n \Delta_j(i))} \mathbf{J}_0 G(x - \tilde{B}_t) \right) \right] \end{aligned}$$

and changing variable  $x - B_t$  to  $x$  in  $\int dx$  we have

$$\begin{aligned} & (F, T_t G) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^0 \left[ \int dx e^{-\int_0^t V(x + B_s) ds} \left( \mathbf{J}_t F(x + B_t), e^{i\sqrt{\alpha} \mathcal{A}_E(\oplus_{j=1}^{d-1} \sum_{i=1}^n \tilde{\Delta}_j(i))} \mathbf{J}_0 G(x) \right) \right], \end{aligned}$$

where

$$\tilde{\Delta}_j(i) = - \sum_{\mu=1}^d \int_{t(i-1)/n}^{ti/n} \mathbf{j}_{t-t(i-1)/n} \rho_\mu^j(x + B_s) \circ dB_s^\mu.$$

and

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \tilde{\Delta}_j(i) = - \sum_{\mu=1}^d \int_0^t \rho_\mu^j(x + B_s) \circ dB_s^\mu.$$

We thus can finally see that

$$(F, T_t G) = \int dx \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} \left( \mathbf{J}_t F(B_t), e^{-i\sqrt{\alpha} \mathcal{A}_E(K_t)} \mathbf{J}_0 G(B_0) \right) \right] = (T_t F, G).$$

Then the lemma follows. QED

**Lemma 2.17** *Suppose Assumption 2.12. Then  $T_t$  is strongly continuous in  $t \geq 0$  on  $\mathcal{H}$ .*

*Proof:* Since  $\|T_t\|$  is uniformly bounded and the semigroup property  $T_t T_s = T_{t+s}$  is hold, it is enough to show the weak continuity at  $t = 0$ . By the Lebesgue dominated convergence theorem it suffices to show that

$$\mathbb{E}^x[(J_0 F(B_0), e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} J_t G(B_t))] \rightarrow \mathbb{E}^x[(J_0 F(B_0), J_0 G(B_0))]$$

as  $t \rightarrow 0$  for each  $x \in \mathbb{R}^d$ . Let

$$\begin{aligned} & \mathbb{E}^x[(J_0 F(B_0), e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} J_t G(B_t))] - \mathbb{E}^x[(J_0 F(B_0), J_0 G(B_0))] \\ &= \mathbb{E}^x[(J_0 F(B_0), e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} J_t G(B_t))] - \mathbb{E}^x[(J_0 F(B_0), e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} J_t G(B_0))] \\ &+ \mathbb{E}^x[(J_0 F(B_0), e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} J_t G(B_0))] - \mathbb{E}^x[(J_0 F(B_0), e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} J_0 G(B_0))] \\ &+ \mathbb{E}^x[(J_0 F(B_0), e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} J_0 G(B_0))] - \mathbb{E}^x[(J_0 F(B_0), J_0 G(B_0))]. \end{aligned}$$

The first and second terms of the right-hand side above converge to zero as  $t \rightarrow 0$ , since  $B_t$  and  $J_t$  are continuous in  $t$ . We will check that the third line also goes to zero. We have

$$\begin{aligned} & \left| \mathbb{E}^x[(J_0 F(B_0), e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} J_0 G(B_0))] - \mathbb{E}^x[(J_0 F(B_0), J_0 G(B_0))] \right| \\ & \leq (\mathbb{E}^x[\|\sqrt{\alpha}\mathcal{A}_E(K_t) J_0 F(B_0)\|^2])^{1/2} (\mathbb{E}^x[\|G(B_t)\|^2])^{1/2} \end{aligned}$$

We have a bound

$$\mathbb{E}^x[\|\mathcal{A}_E(K_t) J_0 F(B_0)\|^2] \leq \|\sqrt{N+1}F(x)\|^2 \mathbb{E}^0[\|K_t(x)\|_{L^2(\mathbb{R}^{d+1})}^2],$$

where  $K_t(x) = \oplus_{j=1}^{d-1} \sum_{\mu=1}^d \int_0^t j_s \rho_\mu^j(x + B_s) \circ dB_s^\mu$ . We have

$$\mathbb{E}^0[\|K_t(x)\|_{L^2(\mathbb{R}^{d+1})}^2] \leq \sum_{j=1}^{d-1} \int_0^t ds \mathbb{E}^x \left[ 2 \sum_{\mu=1}^d \|\rho_\mu^j(B_s)\|^2 + \frac{1}{2} \|\partial \cdot \rho^j(B_s)\|^2 \right]. \quad (2.51)$$

Then  $\lim_{t \rightarrow 0} \mathbb{E}^x[\|\mathcal{A}_E(K_t) J_0 F(B_0)\|^2] = 0$  follows and the proof is complete. QED

**Theorem 2.18** *Suppose Assumption 2.12. Let  $V \in \mathcal{K}$ . Then  $\{T_t\}_{t \geq 0}$  is a strongly continuous one-parameter symmetric semigroup. In particular there exists a self-adjoint operator  $K_{\text{PF}}$  bounded below such that*

$$e^{-tK_{\text{PF}}} = T_t, \quad t \geq 0, \quad (2.52)$$

and

$$e^{-tK_{\text{PF}}} F(x) = \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} J_0^* e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} J_t F(B_t) \right]. \quad (2.53)$$

*Proof:* This follows from Lemmas 2.15, 2.16 and 2.17. QED

**Definition 2.19 (Generalized Pauli-Fierz Hamiltonians)** *Suppose Assumption 2.12. We define a generalized Pauli-Fierz Hamiltonian with an external potential  $V \in \mathcal{K}$  by a self-adjoint operator  $K_{\text{PF}}$  in (2.52).*

**Corollary 2.20** *Suppose Assumption 2.12. Let us identify  $\mathcal{H}$  with  $L^2(\mathbb{R}^d \times Q)$ . Then under this identification  $e^{i(\pi/2)N} e^{-tK_{\text{PF}}} e^{-i(\pi/2)N}$ ,  $t > 0$ , is positivity improving. In particular the ground state of  $K_{\text{PF}}$  is unique if it exists.*

*Proof:* By (2.53) we can see that

$$\begin{aligned} & (F, e^{i(\pi/2)N} e^{-tK_{\text{PF}}} e^{-i(\pi/2)N} G) \\ &= \int dx \mathbb{E}^x \left[ \left( J_0 F(x), e^{-\int_0^t V(B_s) ds} e^{i(\pi/2)N} e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} e^{-i(\pi/2)N} J_t G(B_t) \right) \right]. \end{aligned}$$

Since in [Hir00-a] it is shown that  $e^{i(\pi/2)N} e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} e^{-i(\pi/2)N}$  is positivity improving,  $(F, e^{i(\pi/2)N} e^{-tK_{\text{PF}}} e^{-i(\pi/2)N} G) > 0$  for all  $0 \leq F, G \in \mathcal{H}$  but  $F \neq 0$  and  $G \neq 0$ . Then the corollary follows. QED

Let  $L^p(\mathbb{R}^d; L^2(Q)) = \left\{ f : \mathbb{R}^d \rightarrow L^2(Q) \mid \int \|f(x)\|_{L^2(Q)}^p dx < \infty \right\}$  and set the  $L^p$  norm as  $\|F\|_p = (\int \|F(x)\|_{L^2(Q)}^p dx)^{1/p}$ .

**Corollary 2.21** *Suppose Assumption 2.12.  $e^{-tK_{\text{PF}}}$  can be extended to a bounded operator from  $L^p(\mathbb{R}^d; L^2(Q))$  to itself for  $1 \leq p \leq \infty$ .*

*Proof:* Let  $p \neq \infty$ ,  $p \neq 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Then we have

$$\begin{aligned} \|e^{-tK_{\text{PF}}} F(x)\|_{L^2(Q)}^p &\leq \left( \mathbb{E}^x \left[ e^{-\int_0^t V(B_s) ds} \|F(B_t)\| \right] \right)^p \\ &\leq \left( \mathbb{E}^x \left[ e^{-q \int_0^t V(B_s) ds} \right] \right)^{p/q} \mathbb{E}^x \left[ \|F(B_t)\|_{L^2(Q)}^p \right]. \end{aligned}$$

Thus we have

$$\int \|e^{-tK_{\text{PF}}} F(x)\|_{L^2(Q)}^p dx \leq C \int \|F(x)\|_{L^2(Q)}^p dx.$$

In the case of  $p = \infty$  and  $p = 1$ , the proof is similar. QED

## 2.4 Quadratic form and $K_{\text{PF}}$

By the functional integral representation we have the so-called diamagnetic inequality

$$|(F, e^{-tH_{\text{PF}}}G)| \leq (|F|, e^{-t(H_{\text{p}}+H_{\text{f}}(m))}|G|) \quad (2.54)$$

By means of the diamagnetic inequality we can see that when  $|V|^{1/2}$  is relatively bounded with respect to  $(p^2/2)^{1/2}$  with a relative bound  $a \geq 0$ , it is also relatively bounded with respect to  $(\frac{1}{2}(p + \sqrt{\alpha}\mathcal{A})^2 + H_{\text{f}}(m))^{1/2}$  with a relative bound  $\leq a$ . See [Hir97]. Let  $V = V_+ - V_-$  be such that  $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$  and  $V_-$  infinitesimally small with respect to  $p^2/2$  in the sense of form. Then under Assumption 2.1 we can define the self-adjoint operator

$$H_{\text{PF}} = \frac{1}{2}(p + \sqrt{\alpha}\mathcal{A})^2 + H_{\text{f}}(m) \dot{+} V_+ \dot{-} V_- \quad (2.55)$$

by the quadratic form sum  $\dot{\pm}$ .

**Theorem 2.22** *Let  $V \in \mathcal{K}$  and suppose Assumption 2.1. Then  $K_{\text{PF}} = H_{\text{PF}}$ , where  $H_{\text{PF}}$  is defined by (2.55).*

*Proof:* The functional integral representation of  $e^{-tH_{\text{PF}}}$  for (2.55) can be given by the procedure below [Sim79, Hir97]. Let

$$V_{n,m}(x) = \begin{cases} n, & V(x) \geq n. \\ V(x), & m < V(x) < n, \\ m, & V(x) \leq m. \end{cases}$$

Thus  $V_{n,m} \in L^\infty(\mathbb{R}^d)$  and then the functional integral representation of  $e^{-tH_{\text{PF}}}$  with external potential  $V_{n,m}$ , which is denoted by  $e^{-tH_{\text{PF}}(n,m)}$ , is given by Proposition 2.6. By the monotone convergence theorem for forms, we can see that  $\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} e^{-tH_{\text{PF}}(n,m)} = e^{-tH_{\text{PF}}}$ , where  $H_{\text{PF}}$  is defined by (2.55). On the other hand the functional integral representation of  $I = (F, e^{-tH_{\text{PF}}(n,m)}G) = \Re I + i\Im I$  is divided into the positive part and the negative part as

$$I = (\Re I)_+ - (\Re I)_- + i(\Im I)_+ - i(\Im I)_-,$$

and each term converges as  $n, m \rightarrow \infty$  by the monotone convergence theorem for integral. Then the functional integral representation is given by

$$\begin{aligned} & (F, e^{-tH_{\text{PF}}} G) \\ &= \lim_{n, m \rightarrow \infty} \int dx \mathbb{E} \left[ \left( J_0 F(B_0), e^{-\int_0^t V_{n,+}(B_s) ds} e^{+\int_0^t V_{m,-}(B_s) ds} e^{i\sqrt{\alpha}\mathcal{A}(K_t)} J_t G(B_t) \right) \right]. \\ &= \int dx \mathbb{E} \left[ \left( J_0 F(B_0), e^{-\int_0^t V(B_s) ds} e^{i\sqrt{\alpha}\mathcal{A}(K_t)} J_t G(B_t) \right) \right]. \end{aligned} \quad (2.56)$$

Since  $V \in \mathcal{K}$ , we see that  $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$  and  $V_-$  is infinitesimally small with respect to  $p^2/2$  in the sense of form [CFKS87, Theorem 1.12]. Moreover  $(F, e^{-tK_{\text{PF}}} G)$  equals to the right-hand side of (2.22). Then we conclude that  $e^{-tH_{\text{PF}}} = e^{-tK_{\text{PF}}}$ . Thus the theorem follows. QED

### 3 Pointwise spatial exponential decays

In this section we show the spatial exponential decay of bound states of  $K_{\text{PF}}$ . Let  $\varphi_{\text{b}}$  be a bound state of  $K_{\text{PF}}$  associated with eigenvalue  $E$ ;

$$K_{\text{PF}}\varphi_{\text{b}} = E\varphi_{\text{b}}. \quad (3.1)$$

**Assumption 3.1** *We say that  $V = W + U \in \mathcal{E}$  if and only if  $W \in L^1_{\text{loc}}(\mathbb{R}^d)$ ,  $\inf_x W(x) > -\infty$  and  $0 > U \in L^p(\mathbb{R}^d)$  for some  $p \begin{cases} = 1, & d = 1, \\ > d/2, & d \geq 2. \end{cases}$*

Let  $W + U \in \mathcal{E}$  and set  $W = W_+ - W_-$ , where  $W_{\pm} \geq 0$  is given by  $W_+(x) = \max\{0, W(x)\}$  and  $W_-(x) = \min\{0, W(x)\}$ . Since  $U \in L^p(\mathbb{R}^d) \subset \mathcal{K}_{\text{Kato}}$ ,  $W_- \in L^\infty \subset \mathcal{K}_{\text{Kato}}$  and  $W_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$ , we note that  $\mathcal{E} \subset \mathcal{K}$ . We set

$$W_\infty = \inf_x W(x). \quad (3.2)$$

A fundamental estimate to show the spatial exponential decay of bound states is the lemma below.

**Lemma 3.2** *Let  $V = W + U \in \mathcal{E}$ . Suppose that  $\hat{\rho}_\mu^j \in C^1_{\text{b}}(\mathbb{R}^d_x; L^2(\mathbb{R}^d_k))$ . Then for arbitrary  $t, a > 0$  and each  $0 < \alpha < 1/2$ , there exist constants  $D_1, D_2$  and  $D_3$  such that*

$$\|\varphi_{\text{b}}(x)\|_{L^2(Q)} \leq D_1 e^{D_2 \|U\|_p t} e^{Et} \left( D_3 e^{-\frac{\alpha}{4} \frac{a^2}{t}} e^{-tW_\infty} + e^{-tW_a(x)} \right) \|\varphi_{\text{b}}\|_{\mathcal{H}}, \quad (3.3)$$

where  $W_a(x) = \inf\{W(y) \mid |x - y| < a\}$ .

*Proof:* It is a slight modification of [Car78]. Since  $\varphi_b = e^{tE} e^{-tK_{\text{PF}}} \varphi_b$ , we have

$$\varphi_b(x) = \mathbb{E}^x \left[ J_0^* e^{-\int_0^t V(B_s)} e^{i\sqrt{\alpha}\mathcal{A}_E(K_t)} J_t \varphi_b(B_t) \right] e^{tE}. \quad (3.4)$$

Hence for almost every  $x$  it follows that

$$\|\varphi_b(x)\|_{L^2(Q)} \leq e^{tE} \mathbb{E}^x \left[ e^{-\int_0^t V(B_s)} \|\varphi_b(B_t)\|_{L^2(Q)} \right]. \quad (3.5)$$

By this we have

$$\|\varphi_b(x)\|_{L^2(Q)} \leq e^{tE} \left( \mathbb{E}^x \left[ e^{-4\int_0^t W(B_s)ds} \right] \right)^{1/4} \left( \mathbb{E}^x \left[ e^{-4\int_0^t U(B_s)ds} \right] \right)^{1/4} \|\varphi_b\|_{\mathcal{H}},$$

where we used the Schwartz inequality and

$$\begin{aligned} \mathbb{E}^x [\|\varphi_b(B_t)\|_{L^2(Q)}^2] &= \int (2\pi t)^{-d/2} e^{-|y|^2/2t} \|\varphi_b(x+y)\|_{L^2(Q)}^2 dy \\ &= \int e^{-\pi|z|^2} \|\varphi_b(x + \sqrt{2\pi t}z)\|_{L^2(Q)}^2 dz \leq \|\varphi_b\|_{\mathcal{H}}^2. \end{aligned}$$

Let  $A = \{\omega \in \mathcal{X} \mid \sup_{0 \leq s \leq t} |B_s(\omega)| > a\}$ . Then it follows from a martingale inequality that

$$\mathbb{E}^0[1_A] \leq 2P^0(|B_t| \geq a) = 2(2\pi)^{-d/2} S_{d-1} \int_{a/\sqrt{t}}^{\infty} e^{-r^2/2} r^{d-1} dx \leq \xi_\alpha e^{-\alpha a^2/t}$$

with some  $\xi_\alpha$  for each  $0 < \alpha < 1/2$ . Thus it follows that

$$\begin{aligned} \mathbb{E}^x \left[ e^{-4\int_0^t W(B_s)ds} \right] &= \mathbb{E}^0 \left[ 1_A e^{-4\int_0^t W(B_s+x)ds} \right] + \mathbb{E}^x \left[ 1_{A^c} e^{-4\int_0^t W(B_s)ds} \right] \\ &\leq e^{-4tW_\infty} \mathbb{E}^0[1_A] + e^{-4tW_a(x)} \\ &\leq \xi_\alpha e^{-\alpha a^2/t} e^{-4tW_\infty} + e^{-4tW_a(x)}. \end{aligned}$$

Next we estimate  $\mathbb{E}^x \left[ e^{-4\int_0^t U(B_s)ds} \right]$ . Since  $U$  is in Kato-class, there exist constants  $D_1$  and  $D_2$  such that  $\mathbb{E}^x \left[ e^{-4\int_0^t U(B_s)ds} \right] \leq D_1 e^{D_2 \|U\|_p t}$  by Lemmas 2.11. Setting  $D_3 = \xi_\alpha^{1/4}$ , we obtain the lemma by the inequality  $(a+b)^{1/4} \leq a^{1/4} + b^{1/4}$  for  $a, b \geq 0$ . QED

For  $V = W + U \in \mathcal{E}$ , we define

$$\Sigma = \liminf_{|x| \rightarrow \infty} V(x). \quad (3.6)$$



Since  $U \in L^p(\mathbb{R}^d)$ ,  $\liminf_{|x| \rightarrow \infty} U(x) = 0$  and hence

$$\Sigma = \liminf_{|x| \rightarrow \infty} W(x). \quad (3.7)$$

Moreover  $\Sigma \geq W_\infty$  holds.

**Theorem 3.3** *Suppose that  $V = W + U \in \mathcal{E}$  and  $\hat{\rho}_\mu^j \in C_b^1(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$ .*

**(Confining case 1)** *Suppose that  $W(x) \geq \gamma|x|^{2n}$  outside a compact set  $K$  for some  $n > 0$  and some  $\gamma > 0$ . Let  $0 < \alpha < 1/2$ . Then there exists a constant  $C_1$  such that*

$$\|\varphi_b(x)\|_{L^2(Q)} \leq C_1 \exp\left(-\frac{\alpha c}{16}|x|^{n+1}\right) \|\varphi_b\|_{\mathcal{H}}, \quad (3.8)$$

where  $c = \inf_{x \in \mathbb{R}^d \setminus K} W_{\frac{|x|}{2}}(x)/|x|^{2n}$ .

**(Confining case 2)** *Suppose that  $\lim_{|x| \rightarrow \infty} W(x) = \infty$ . Then there exist constants  $C$  and  $\delta$  such that*

$$\|\varphi_b(x)\|_{L^2(Q)} \leq C \exp(-\delta|x|) \|\varphi_b\|_{\mathcal{H}}. \quad (3.9)$$

**(Non-confining case)** *Suppose that  $\Sigma > E$  and  $\Sigma > W_\infty$ . Let  $0 < \beta < 1$ . Then there exists a constant  $C_2$  such that*

$$\|\varphi_b(x)\|_{L^2(Q)} \leq C_2 \exp\left(-\frac{\beta}{8\sqrt{2}} \frac{(\Sigma - E)}{\sqrt{\Sigma - W_\infty}} |x|\right) \|\varphi_b\|_{\mathcal{H}}. \quad (3.10)$$

*Proof:* Since  $\sup_x \|\varphi_b(x)\|_{L^2(Q)} < \infty$ , it is enough to show all the statements for sufficiently large  $|x|$ .

(Confining case 1) Note that  $W_{\frac{|x|}{2}}(x) \geq c|x|^{2n}$  for  $x \in \mathbb{R}^d \setminus K$ . Then we have bounds for  $x \in \mathbb{R}^d \setminus K$ :

$$|x|W_{\frac{|x|}{2}}(x)^{1/2} \geq c|x|^{n+1}, \quad (3.11)$$

$$|x|W_{\frac{|x|}{2}}(x)^{-1/2} \leq c|x|^{1-n}. \quad (3.12)$$

Inserting  $t = t(x) = W_{\frac{|x|}{2}}(x)^{-1/2}|x|$  and  $a = a(x) = \frac{|x|}{2}$  in (3.3), we have

$$\|\varphi_b(x)\| \leq e^{-\frac{\alpha}{16}c|x|^{n+1}} D_1 e^{(D_2\|U\|_p + E)c|x|^{1-n}} \left( D_3 e^{c|x|^{1-n}|W_\infty|} + e^{-(1-\frac{\alpha}{16})c|x|^{n+1}} \right) \|\varphi_b\|_{\mathcal{H}} \quad (3.13)$$

for  $x \in \mathbb{R}^d \setminus K$ . Then (3.8) follows.

(Non-confining case) Rewrite formula (3.3) as

$$\|\varphi_b(x)\| \leq D_1 e^{D_2 \|U\|_p t} \left( D_3 e^{-\frac{\alpha}{4} \frac{a^2}{t}} e^{-t(W_\infty - E)} + e^{-t(W_a(x) - E)} \right) \|\varphi_b\|_{\mathcal{H}}. \quad (3.14)$$

Then altering both  $\Sigma = \liminf_{|x| \rightarrow \infty} (-W_-(x))$  and  $\Sigma > W_\infty$ , it is possible to choose decomposition  $V = W + U \in \mathcal{E}$  such that  $\|U\|_p \leq (\Sigma - E)/2$ , since  $\liminf_{|x| \rightarrow \infty} U(x) = 0$ . Inserting  $t = t(x) = \epsilon|x|$  and  $a = a(x) = \frac{|x|}{2}$  in (3.14), we have

$$\begin{aligned} \|\varphi_b(x)\| &\leq D_1 e^{\|U\|_p \epsilon |x|} \left( D_3 e^{-\frac{\alpha}{16\epsilon} |x|} e^{-\epsilon |x| (W_\infty - E)} + e^{-\epsilon |x| (W_{\frac{|x|}{2}}(x) - E)} \right) \|\varphi_b\|_{\mathcal{H}} \\ &\leq D_1 \left( D_3 e^{-\left(\frac{\alpha}{16\epsilon} + \epsilon(W_\infty - E) - \frac{1}{2}\epsilon(\Sigma - E)\right)|x|} + e^{-\epsilon \left(W_{\frac{|x|}{2}}(x) - E - \frac{1}{2}(\Sigma - E)\right)|x|} \right) \|\varphi_b\|_{\mathcal{H}}. \end{aligned}$$

Choosing  $\epsilon = \frac{\sqrt{\alpha/16}}{\sqrt{\Sigma - W_\infty}}$ , the exponent on the first term above turns out to be

$$\frac{\alpha}{16\epsilon} + \epsilon(W_\infty - E) - \frac{1}{2}\epsilon(\Sigma - E) = \frac{1}{2}\epsilon(\Sigma - E).$$

Moreover we see that  $\liminf_{|x| \rightarrow \infty} W_{\frac{|x|}{2}}(x) = \Sigma$ , and obtain

$$\|\varphi_b(x)\|_{L^2(Q)} \leq C_2 e^{-\frac{\epsilon}{2}(\Sigma - E)|x|} \|\varphi_b\|_{\mathcal{H}}$$

for sufficiently large  $|x|$ . Then (3.10) follows.

(Confining case 2) Finally we prove confining case 2. In this case for arbitrary  $c > 0$  there exists  $N$  such that  $W_{\frac{|x|}{2}}(x) \geq c$  for all  $|x| > N$ . Inserting  $t = t(x) = \epsilon|x|$  and  $a = a(x) = \frac{|x|}{2}$  in (3.3), we obtain that

$$\begin{aligned} \|\varphi_b(x)\| &\leq D_1 e^{\|U\|_p \epsilon |x|} \left( D_3 e^{-\frac{\alpha}{16\epsilon} |x|} e^{-\epsilon |x| (W_\infty - E)} + e^{-\epsilon |x| (W_{\frac{|x|}{2}}(x) - E)} \right) \|\varphi_b\|_{\mathcal{H}} \\ &\leq D_1 \left( D_3 e^{-\left(\frac{\alpha}{16\epsilon} - \epsilon\|U\|_p + \epsilon(W_\infty - E)\right)|x|} + e^{-\epsilon |x| (c - E - \|U\|_p)} \right) \|\varphi_b\|_{\mathcal{H}} \end{aligned}$$

for  $|x| > N$ . Choosing sufficiently large  $c$  and sufficiently small  $\epsilon$  such that

$$\begin{aligned} \frac{\alpha}{16\epsilon} - \epsilon\|U\|_p + \epsilon(W_\infty - E) &> 0, \\ c - E - \|U\|_p &> 0, \end{aligned}$$

we have  $\|\varphi_b(x)\| \leq C'e^{-\delta'|x|}$  for sufficiently large  $|x|$ . Then (3.9) follows. QED

We give several remarks on Theorem 3.3.

**(Independence of bose mass  $m$ )** Suppose that  $\omega(k) = \sqrt{|k|^2 + m^2}$ . Let  $\varphi_b$  be a normalized ground state of  $K_{\text{PF}}$ :  $\|\varphi_b\|_{\mathcal{H}} = 1$ , and  $E_m = \inf \sigma(K_{\text{PF}})$ . It is shown that there exist also constants  $C_1$  and  $C_2$  such that

$$\|\varphi_b(x)\|_{L^2(Q)} \leq C_1 e^{-C_2|x|^n}, \quad n \geq 1,$$

by Theorem 3.3. Since the ground state energy  $E_m$  is decreasing in  $m$ , we can take  $C_1$  and  $C_2$  independent of  $m < M$  with some  $M$ . This fact is nontrivial and useful to show the existence of ground states of the Pauli-Fierz model with  $m = 0$ . This is used in e.g., [Hid10].

**(Condition  $W_\infty < \Sigma$ )** When  $\inf_x V(x) < \Sigma$ , it is possible to decompose  $V = W + U \in \mathcal{E}$  such that  $W_\infty < \Sigma$ . In fact for arbitrary  $\epsilon > 0$ , there exists  $y \in \mathbb{R}^d$  such that

$$V(y) < \inf_x V(x) + \epsilon.$$

Suppose that  $\inf_x V(x) + \epsilon < \Sigma$ . Let  $\mathcal{O}_y \subset \mathbb{R}^d$  be a neighborhood of  $y$ . Then define  $u(x) = \begin{cases} U(x), & x \in \mathcal{O}_y, \\ 0, & y \notin \mathcal{O}_y. \end{cases}$  Let  $\tilde{W} = W + u$  and  $\tilde{U} = U - u$ . This yields that  $V = \tilde{W} + \tilde{U} \in \mathcal{E}$  and  $\tilde{W}_\infty < \inf_x V(x) + \epsilon < \Sigma$ .

**(Threshold)** The threshold is defined by

$$\Sigma_\infty = \lim_{R \rightarrow \infty} \inf_{F \in D_R, \|F\|=1} (F, H_{\text{PF}} F),$$

where  $D_R = \{F \in D(H_{\text{PF}}) | F(x) = 0, |x| < R\}$ . We note that  $\Sigma_\infty \geq \Sigma$ , and  $\Sigma = \Sigma_\infty = \infty$  in confining cases.

The bound given in [Gri01] is  $\|e^{+C|\cdot|} 1_{(-\infty, \lambda]}(H_{\text{PF}})\|_{\mathcal{H}} < \infty$ , where  $C^2 + \lambda < \Sigma_\infty$ . From this the bound

$$\int dx \|e^{+\delta|x|} \varphi_b(x)\|_{L^2(Q)}^2 \leq C' \|\varphi_b\|_{\mathcal{H}} \quad (3.15)$$

follows, where

$$\delta < \sqrt{\Sigma_\infty - E}.$$

Theorem 3.3, however, gives *pointwise* bounds:

$$\|\varphi_b(x)\|_{L^2(Q)} \leq C_1 \exp(-C_2|x|^\beta) \|\varphi_b\|_{\mathcal{H}}, \quad \beta \geq 1. \quad (3.16)$$

In particular the superexponential decay,  $\|\varphi_b(x)\| \leq C_1 e^{-C_2|x|^{n+1}} \|\varphi_b\|_{\mathcal{H}}$ , is shown for the case of polynomially increasing potentials (confining case 1), while in non-confining cases, we show that in (3.16),  $\beta = 1$  and

$$C_2 < \frac{\Sigma - E}{8\sqrt{2}\sqrt{E - W_\infty}}. \quad (3.17)$$

We give examples of external potentials.

**Example 3.4 (Confining potentials)** Let  $V = V_+ - V_-$  be such that  $V_+ \in L^p_{\text{loc}}(\mathbb{R}^d)$  and  $V_- \in L^p(\mathbb{R}^d)$ , where  $p \begin{cases} = 1, & d = 1, \\ > d/2, & d \geq 2. \end{cases}$  In this case  $V \in \mathcal{E}$ .

**Example 3.5 (Coulomb potentials)** Suppose Assumption 2.1. Then

$$H_{\text{PF}} = K_{\text{PF}}.$$

Let  $V = -\alpha Z/|x|$  be the Coulomb potential. Then  $\inf \sigma(H_p) = -\alpha Z/2$ . We have  $(\phi \otimes 1, H_{\text{PF}} \phi \otimes 1)_{\mathcal{H}} = (\phi, (H_p + V_{\text{eff}})\phi)_{L^2(\mathbb{R}^d)}$  for  $\phi \in D(\frac{1}{2}p^2)$ , where

$$V_{\text{eff}}(x) = \frac{\alpha}{2} \sum_{j=1}^{d-1} \sum_{\mu, \nu=1}^d (\rho_\mu^j(x), \rho_\nu^j(x))_{L^2(\mathbb{R}^d)}.$$

Let  $V_\infty = \sup_x |\sum_{j=1}^{d-1} \sum_{\mu, \nu=1}^d (\rho_\mu^j(x), \rho_\nu^j(x))_{L^2(\mathbb{R}^d)}|$ . Thus

$$\inf \sigma(H_{\text{PF}}) \leq -\frac{\alpha}{2}(Z - V_\infty).$$

When  $Z > V_\infty$ ,  $\inf \sigma(H_{\text{PF}}) < \lim_{|x| \rightarrow \infty} V(x) = 0$  follows for all values of coupling constant  $\alpha$ . Then ground states of  $H_{\text{PF}}$  decay as  $C_1 e^{-C_2|x|}$  pointwise for all values of coupling constants.

## 4 Appendix

In this appendix we show the unitary equivalence between  $H_{\text{PF}}$  and the Pauli-Fierz Hamiltonian defined on

$$L^2(\mathbb{R}^d) \otimes \mathcal{F},$$

where  $\mathcal{F} = \bigoplus_{n=0}^{\infty} \otimes_s^n (\bigoplus^{d-1} L^2(\mathbb{R}^d))$  is the Boson Fock space over  $\bigoplus^{d-1} L^2(\mathbb{R}^d)$ . Let  $\Omega = \{1, 0, 0, \dots\} \in \mathcal{F}$  be the Fock vacuum. The annihilation operator and the creation operator in  $\mathcal{F}$  are denoted by  $a^*(f)$  and  $a(f)$ , respectively, where  $f = (f_1, \dots, f_{d-1}) \in \bigoplus^{d-1} L^2(\mathbb{R}^d)$ . They satisfy canonical commutation relations:

$$\begin{aligned} [a(f), a^*(g)] &= \sum_{j=1}^{d-1} (\bar{f}_j, g_j)_{L^2(\mathbb{R}^d)}, \\ [a^*(f), a^*(g)] &= 0 = [a(f), a(g)]. \end{aligned}$$

The field operator in  $\mathcal{F}$  is given by

$$A(\hat{\phi}) = \frac{1}{\sqrt{2}}(a^*(\hat{\phi}) + a(\tilde{\hat{\phi}})),$$

where  $\tilde{\hat{\phi}}(k) = \hat{\phi}(-k)$ . The quantized radiation field is defined by  $A_\mu = \int_{\mathbb{R}^d}^\oplus A_\mu(x) dx$  under the identification  $L^2(\mathbb{R}^d) \otimes \mathcal{F} \cong L^2(\mathbb{R}^d; \mathcal{F})$  and  $A_\mu(x) = A(\hat{\rho}_\mu(x))$ , where a cutoff function is given by  $\hat{\rho}_\mu(x) = \hat{\rho}_\mu(k, x) = \bigoplus_{j=1}^{d-1} \hat{\phi}_\mu^j(k) \bar{\Psi}(k, x) / \sqrt{\omega(k)}$ . Finally the free field Hamiltonian is defined by

$$d\Gamma(\omega) = \bigoplus_{k=0}^{\infty} \sum_{i=1}^k \underbrace{1 \otimes \dots \otimes \overset{i}{\omega} \otimes \dots \otimes 1}_k. \quad (4.1)$$

Then the Pauli-Fierz Hamiltonian in  $L^2(\mathbb{R}^d) \otimes \mathcal{F}$  is given by

$$\hat{H}_{\text{PF}} = \frac{1}{2}(p \otimes 1 + \sqrt{\alpha} A)^2 + V \otimes 1 + 1 \otimes d\Gamma(\omega). \quad (4.2)$$

Suppose that  $V$  is relatively bounded with respect to  $\frac{1}{2}p^2$  with a relative bound strictly smaller than one, and that  $\hat{\rho}_\mu^j \in C_b^1(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d))$  and

$$\omega \hat{\rho}_\mu^j, \hat{\rho}_\mu^j, \hat{\rho}_\mu^j / \sqrt{\omega}, \partial_{x_\mu} \hat{\rho}_\mu^j, \partial_{x_\mu} \hat{\rho}_\mu^j / \sqrt{\omega} \in L^\infty(\mathbb{R}_x^d; L^2(\mathbb{R}_k^d)). \quad (4.3)$$

See Assumption 2.1. Then  $\hat{H}_{\text{PF}}$  is self-adjoint on  $D(p^2 \otimes 1) \cap D(1 \otimes d\Gamma(\omega))$ . Now let us see the relationship between  $L^2(Q)$  and  $\mathcal{F}$ . Let  $\mathcal{U} : \mathcal{F} \rightarrow L^2(Q)$  be defined by

$$\begin{aligned} \mathcal{U}\Omega &= 1, \\ \mathcal{U} : A(\hat{\phi}_1) \cdots A(\hat{\phi}_n) : \Omega &=: \mathcal{A}(\phi_1) \cdots \mathcal{A}(\phi_n) :, \end{aligned}$$

where the Wick product on the left hand side is defined by moving all the creation operators to the left and annihilation operators to the right without any commutation relations. While the Wick product of the left hand side is defined recursively by

$$: \mathcal{A}(\phi) := \mathcal{A}(\phi)$$

and

$$: \mathcal{A}(\phi) \prod_{j=1}^n \mathcal{A}(\phi_j) := \mathcal{A}(\phi) : \prod_{j=1}^n \mathcal{A}(\phi_j) : - \frac{1}{2} \sum_{k=1}^n (f_k, f) : \prod_{j \neq k} \mathcal{A}(\phi_j) : .$$

The unitary operator  $\mathcal{U}$  can be extended to the unitary operator from  $\mathcal{F}$  to  $L^2(Q)$ , and it also implements

$$\mathcal{U} d\Gamma(\omega) \mathcal{U}^{-1} = H_f(m).$$

Then under (4.3) it follows that  $(1 \otimes \mathcal{U})$  maps  $D(\frac{1}{2}p^2 \otimes 1) \cap D(1 \otimes d\Gamma(\omega))$  to  $D(\frac{1}{2}p^2 \otimes 1) \cap D(1 \otimes H_f(m))$  and

$$(1 \otimes \mathcal{U}) \hat{H}_{\text{PF}} (1 \otimes \mathcal{U}^{-1}) = H_{\text{PF}}. \quad (4.4)$$

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